Some Remarks on Landau-Siegel Zeros

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February 14, 2024

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Motivation and Background

Landau-Siegel Zeros

- Dirichlet (1837) introduced characters $\chi(\text{mod}\,D)$ to prove there are infinitely many primes $p \equiv a \pmod D, (a, D) = 1$. One key step in the proof is to show that $L(1, \chi) \neq 0$ for each non-principal character $\chi(\text{mod}\,D)$.

- Dirichlet (1837) introduced characters $\chi(\bmod D)$ to prove there are infinitely many primes $p \equiv a \,(\bmod D), (a, D) = 1$. One key step in the proof is to show that $L(1, \chi) \neq 0$ for each non-principal character $\chi(\bmod D)$.
- Fairly easy to show that $L(1, \chi) \neq 0$ if χ is complex (so that $\bar{\chi} \neq \chi$), but the non-vanishing of $L(1, \chi)$ for real characters χ is more subtle.

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- To this end, Dirichlet developed his class number formula

$$L(1,\chi_D) = \frac{\pi h(-D)}{\sqrt{D}}, \quad D > 4.$$

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  with effective constant.
- For some applications, lower bound $L(1, \chi) \gg D^{-1/2}$ is not strong enough.

- Assuming GRH:

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- Unconditionally, can show $L(1,\chi) \ll \log D$, but lower bounds are more difficult to obtain.

- Not able to rule out a real zero β of $L(s,\chi)$ with β close to $s = 1$. Such a real zero β is a Landau-Siegel zero.

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- Landau showed that exceptional characters, if they exist, appear only rarely.

- Hecke showed that no real zero in classical zero-free region implies

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- In such a situation, this yields a respectable bound

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- By Hecke's result it follows that $L(1,\chi) = o\left((\log D)^{-1}\right) \implies L(s,\chi)$ has a Landau-Siegel zero.

- One can obtain stronger lower bounds on $L(1, \chi)$.

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- For instance, Landau (1935) showed

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- Unfortunately, the proofs, in principle, doesn't allow for a determination of the constant in terms of ε.

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- Assuming strong Landau-Siegel zero (i.e. $L(1, \chi)$ very small), Friedlander-Iwaniec (2003) have shown in certain ranges that $L < 2 - \frac{1}{59}$.

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- GRH gives $L < 2 + \varepsilon$.

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- However, since we do not believe Landau-Siegel zeros exist, we think of these results as being “illusory”.
- They look impressive, but they lose content if such zeros are finally eliminated.
- Why prove illusory results?

- Some results require considering separately the case where a Landau-Siegel zero exists, and the case where it does not, for e.g., Linnik's Theorem.

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- Girolamo Saccheri in his Euclides Vindicatus (1733) essentially discovered Hyperbolic Geometry, by building around the hypothesis that the angles of a triangle add up less than 180°.

- There are infinitely many prime pairs $p, p+h$ for any fixed nonzero h (Heath-Brown, 1983).

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- Primes in the short interval $(x-y, x]$ for any $y > x^{1/2-1/58+\varepsilon}$ (Friedlander-Iwaniec, 2004). RH gives $y > x^{1/2+\varepsilon}$.
- Infinitely many primes of the form $p = a^6 + b^2$ (Friedlander-Iwaniec, 2005).

Consequences of Landau-Siegel Zeros

Some Illusory Results

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- The Hardy-Littlewood Chowla Conjecture (Tao-Teräväinen, 2021): For $0 \leq k \leq 2$ and $\ell \geq 0$ and any distinct integers $h_1, \ldots, h_k, h'_1, \ldots, h'_\ell$, an asymptotic formula for

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- Assuming the existence of an exceptional character mod D, Čech and Matomäki showed non-vanishing of $L(1/2, \chi)$ for almost all $\chi (\bmod q)$, for any $q \in [D^{300}, D^{O(1)}]$. On GRH one can show 50% of central values are non-vanishing.

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- Heuristic for why this is true:

$$L(1,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1},$$

so if LHS is small, we must have $\chi(p) = -1$ for many p on RHS.

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  so if LHS is small, we must have $\chi(p) = -1$ for many p on RHS.
- This implies $\mu(n) \approx \chi(n)$ for squarefree n. This is a powerful piece of information.

- Define

$$\mathcal{S} = \{\chi \,(\mathrm{mod}\, q) : \chi \text{ primitive and real } \}.$$

Refinements of Siegel's Theorem

Tatuzawa's Result

- Define
$$\mathcal{S} = \{\chi \pmod q : \chi \text{ primitive and real } \}.$$

- It follows from a minor modification of Tatuzawa's refinement of Siegel's theorem that for all $0 < \varepsilon < 1/2$, there exists effectively computable constants $q_0 = q_0(\varepsilon) > 0$ such that
$$\#\left\{\chi \in \mathcal{S} : q \geq q_0 \text{ and } L(s,\chi) \text{ has a real zero in } \left[1 - q^{-\varepsilon}, 1\right)\right\} \leq 1.$$

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- Further numerical refinements of Tatuzawa's result due to Hoffstein, Ji-Lu and Chen.

- Sarnak and Zaharescu improved Tatuzawa's theorem assuming that if $\nu \in \mathcal{S}$ and ω is a zero of $L(s,\nu)$, then $\mathrm{Re}(\omega) = \frac{1}{2}$ or $\mathrm{Im}(\omega) = 0$.

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- Subject to this hypothesis, it follows from their work that for any $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\varepsilon) > 0$ such that

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- In particular, they "exponentiate" the quality of the zero free region at the cost of a hypothesis that, while assuming the generalized Riemann hypothesis for the non-real zeros, still permits the existence of Landau-Siegel zeros.

- We prove that the conclusion of Sarnak and Zaharescu holds under a significantly weaker hypothesis. Fix $0 < \delta < 1/2$.

Hypothesis (H_δ)

If $\nu \in \mathcal{S}$, then all the zeros of $L(s, \nu)$ in the disk $|z - 1| < \delta$ are real.

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Theorem (B.-Thorner-Zaharescu)

Fix $0 < \delta \leq 1/2$. Assume that H_δ is true. For any $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\delta, \varepsilon) > 0$ such that

$$\#\left\{\chi \in \mathcal{S} : q \geq q_0 \text{ and } L(s,\chi) \text{ has a real zero in } \left[1 - (\log q)^{-\varepsilon}, 1\right)\right\} \leq 1.$$

- For $q_0 = q_0(\delta, \varepsilon)$ to be optimized, suppose there exists χ_1 and χ_2 of conductors $q_1 \geqslant q_0$ and $q_2 \geqslant q_0$, such that $L(s, \chi_1)$ and $L(s, \chi_2)$ have real zeros β_1 and β_2 respectively satisfying

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- Define $F(s) = \zeta(s)L(s, \chi_1)\, L(s, \chi_2) L(s, \chi_1\chi_2)$. For $\mathrm{Re}(s) > 1$,

$$-\frac{F'}{F}(s) = \sum_{n \geq 1} \frac{\Lambda(n)\,(1 + \chi_1(n))\,(1 + \chi_2(n))}{n^s}.$$

Refinements of Siegel's Theorem

A Brief sketch of the Proof : Non-Negativity

- For $q_0 = q_0(\delta, \varepsilon)$ to be optimized, suppose there exists χ_1 and χ_2 of conductors $q_1 \geqslant q_0$ and $q_2 \geqslant q_0$, such that $L(s, \chi_1)$ and $L(s, \chi_2)$ have real zeros β_1 and β_2 respectively satisfying

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- On the other hand, we have the partial fraction expansion

$$-\frac{F'}{F}(s) = \frac{1}{s-1} - \sum_{F(\rho)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + B.$$

- Differentiating $(kl-1)$ times, we obtain

$$\frac{1}{(s-1)^{k\ell}} - \sum_{F(\rho)=0} \frac{1}{(s-\rho)^{k\ell}} = \frac{1}{(k\ell-1)!} \sum_{n \geqslant 1} \frac{\Lambda(n)\,(1+\chi_1(n))\,(1+\chi_2(n))\,(\log n)^{k\ell-1}}{n^s}.$$

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- Choose $s = 1+\eta$, for some $\eta > 0$ that we will optimize. Using non-negativity and taking the real part, one has

$$\frac{1}{\eta^{k\ell}} - \mathrm{Re} \sum_{F(\rho)=0} \frac{1}{(1+\eta-\rho)^{k\ell}} \geqslant 0$$

- Rearranging, we arrive at

$$\frac{1}{\eta^{k\ell}} - \frac{1}{(1+\eta-\beta_1)^{k\ell}} \geqslant \frac{1}{(1+\eta-\beta_2)^{k\ell}} + \mathrm{Re} \sum_{\substack{F(\rho)=0 \\ \mathrm{Im}\,\rho\neq 0}} \frac{1}{(1+\eta-\rho)^{k\ell}}. \tag{1}$$

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- Hypothesis H_δ says complex zeros of $F(s)$ cannot come very close to $s = 1+\eta$. So the RHS of (1) is dominated by the real zero β_2.
- Applying Turan's Inequality, for some $r = O(1)$, one has

$$\frac{1}{\eta^{r\ell}} - \frac{1}{(1+\eta-\beta_1)^{r\ell}} \geqslant \frac{1}{8(1+\eta-\beta_2)^{r\ell}}.$$

- By optimizing η and q_0 in terms of δ and ε, we arrive at a contradiction.

Thank you for your attention!