Automorphic Forms and Representations Extended Syllabus

UIUC Graduate Number Theory Seminar

March 2024

The goal of this reading group will be to understand the definition of Automorphic Forms and Automorphic Representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and how it relates to the classical theory of Automorphic Forms for GL_2. The main references will be Goldfeld and Hundley's book, *Automorphic Representations and L-Functions for the General Linear Group* [GH]. The suggested exercises are all taken from [GH] as well.

1 Adeles

Introduce the definition of adeles and ideles of global fields, their algebraic/topological properties. Introduce integration on the p-adics and the adeles. Define the Fourier transform on the p-adics and the adeles, and Bruhat-Schwartz functions on $\mathbb{A}_{\mathbb{Q}}$. Use these to state Adelic Poisson summation, explaining the relationship between it and usual Poisson summation. References: [GH] Chapter 1. Suggested Exercises: 1.6, 1.10, 1.12, 1.13, 1.14

2 Automorphic forms and L-functions for $GL_1(\mathbb{A}_{\mathbb{Q}})$

Define Hecke characters (quasicharacter) and unitary Hecke characters for global fields and explain their relationship. Introduce automorphic forms for $GL_1(\mathbb{A}_{\mathbb{Q}})$ and explain how classical Dirichlet characters show up. Define local zeta functions and show that they satisfy a functional equation. Show that the local factors combine to give you a functional equation for the completed L-function and that the local root numbers multiply to give you the global Artin root number. References: [GH] Sections 2.1-2.4. Suggested Exercises: 2.1, 2.6, 2.8

3 Automorphic representations for $GL_1(\mathbb{A}_{\mathbb{Q}})$

Define automorphic representations for $GL_1(\mathbb{A}_{\mathbb{Q}})$ and show that they are factorizable. How does an automorphic representation "come from" an automorphic form? Explain what Hecke operators on $GL_1(\mathbb{A}_{\mathbb{Q}})$ are and why they aren't interesting. If time permits, explain the Rankin-Selberg method and its adelic analogue for $GL_1(\mathbb{A}_{\mathbb{Q}})$. References: [GH] Sections 2.5-2.8. Suggested Exercises: 2.9, 2.10, 2.11, 2.12, 2.13a

4 Review of Classical Automorphic Forms for GL_2

Define a classical automorphic form (automorphic function) of integral weight k and character ψ on congruence subgroups of $SL_2(\mathbb{Z})$. Introduce Maass forms, their Fourier-Whittaker expansions, and Maass cusp forms. Use Maass lowering/raising to show that the bottom of the spectrum of Laplace operators Δ_k has as eigenfunctions the classical modular forms (times some power of y). Define Hecke operators on the space of classical automorphic forms. State the Ramanujan-Petersson conjecture and its generalization to Maass Forms. [GH] Chapter 3, 11. Suggested Exercises: 3.3, 3.12, 3.13, 3.14, 3.18, 3.19

5 Interlude into matrix decompositions and Lie Algebra Theory for $\mathfrak{gl}_2(\mathbb{R}), \mathfrak{gl}_2(\mathbb{C})$

Introduce the Iwasawa and Cartan decompositions for $GL_2(\mathbb{R}), GL_2(\mathbb{Q}_p), GL_2(\mathbb{A}_{\mathbb{Q}})$. If time permits, explain them for connected semisimple real Lie groups. Define the universal enveloping algebra of a Lie algebra, particularly for $\mathfrak{gl}_2(\mathbb{R}), \mathfrak{gl}_2(\mathbb{C})$ and why we care. References: [GH] Sections 4.1-4.6. Suggested Exercises: 4.2, 4.4, 4.6, 4.13

6 Automorphic Forms for $GL_2(\mathbb{A}_\mathbb{Q})$

Define automorphic Forms for $GL_2(\mathbb{A}_\mathbb{Q})$. Carefully prove that even weight zero (classical) Maass Forms for $SL_2(\mathbb{Z})$ "lift" to automorphic forms for $GL_2(\mathbb{A}_\mathbb{Q})$. Generalize to arbitrary (integral) weight, level, character. References: [GH] Sections 4.7-4.12. Suggested Exercises: 4.16, 4.17, 4.20, 4.21

7 Automorphic Representations for $GL_2(\mathbb{A}_\mathbb{Q})$

Explain the actions of the finite adeles, the orthogonal group, and the universal enveloping algebra on the space of automorphic forms for $GL_2(\mathbb{A}_\mathbb{Q})$ with prescribed Hecke character. Point out where we use a nontrivial theorem of Harish-Chandra to show the action is well-defined. Define $(\mathfrak{g}, K_\infty)$-modules and $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-modules. Finally, define automorphic representations and cuspidal automorphic representations. Work through the example of obtaining a cuspidal automorphic representation from weight k new Maass forms with character for the congruence subgroup $\Gamma_0(N)$. References:[GH] Sections 5.1-5.4. Suggested Exercises: 5.2, 5.5, 5.8

8 Admissible $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-modules

Define admissible $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-modules, and show that irreducible cuspidal automorphic representations are admissible. Show that we can obtain a classical automorphic form for GL_2 from an automorphic form for $GL_2(\mathbb{A}_\mathbb{Q})$. References:[GH] Section 5.5. Suggested Exercises: 5.15, 5.22

9 Automorphic L-functions for $GL_2(\mathbb{A}_\mathbb{Q})$

Explain how irreducible admissible automorphic representations for $GL_2(\mathbb{A}_\mathbb{Q})$ is a restricted tensor product of irreducible admissible representations at the local factors, and the local representations are unique up to equivalence. Describe the contragredient representation of an automorphic representation. Define L-functions associated to local automorphic representations, and construct the L-function of the automorphic representation we started with, taking note of the functional equation it satisfies. If time permits, discuss converse theorems and the general conjecture of Cogdell-Piatetski-Shapiro. State the Ramanujan conjecture for global cuspidal automorphic representations. References: [Gel] Section 2, [GH] Chapters 9-11, [JL] Sections 2,9,11, [Tay] Section 3

10 Automorphic Forms beyond GL_2

Optional section. Define admissible representations of nonarchimedean and archimedean Hecke algebras of an algebraic group G over the ring of integers O_F of a global field so that G_F is reductive. Define automorphic representations of $G(\mathbb{A}_F)$ and automorphic forms for arithmetic subgroups of $G(F)$ and $G(\mathbb{A}_F)$. Show how they specialize to classical automorphic forms for GL_2 and automorphic forms for $GL_2(\mathbb{A}_\mathbb{Q})$. References:[Get] Sections 3.1-3.4 [1]

References

[Gel] Stephen Gelbart. "Three Lectures on the Modularity of $\rho_{E,3}$ and the Langlands Reciprocity Conjecture". In: *Modular Forms and Fermat's Last Theorem*. Ed. by Gary Cornell, Joseph H. Silverman, and Glenn Stevens. New York, NY: Springer New York, 1997, pp. 155–207. ISBN: 978-1-4612-1974-3. DOI: 10.1007/978-1-4612-1974-3_6. URL: https://doi.org/10.1007/978-1-4612-1974-3_6.

[Get] Jayce R. Getz. *An introduction to Automorphic Representations*. URL: https://services.math.duke.edu/~jgetz/aut_reps.pdf.

[GH] Dorian Goldfeld and Joseph Hundley. *Automorphic Representations and L-Functions for the General Linear Group*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

[JL] R. P. Langlands H. Jacquet. *Automorphic Forms on GL (2). Part 1*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1970.

[Tay] Richard Taylor. "Galois Representations". In: *Annales de la Faculte des Sciences de Toulouse 13* (2004), pp. 73–119.

[1] Title taken from Arizona Winter School 2022